Synchrotron emission and introduction to receivers

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1 Derivation of synchrotron power spectrum

Synchrotron emission is radiation from relativistic, charged particles moving in a magnetic field. The equation of motion of such a charged particle is

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m \vec{v}\right) = \frac{q}{c}\vec{v} \times \vec{B} \tag{1}$$

in cgs units. We neglect $\vec{E}$ in most scenarios in astrophysics because, if there were an electric field, free charges would immediately move to neutralize it. The motion of the charged particle is helical around the magnetic field line with

$$\omega_B = \frac{qB}{\gamma m c} \tag{2}$$

as the frequency of gyration.

We go to the rest frame of the particle, where the acceleration is nonzero, find the emitted power, and transform to the lab frame. Use Larmor's formula for a slowly-moving particle:

$$P = \frac{2q^2}{3c^3}\left|\ddot{\vec{r}}\right|^2 \tag{3}$$

In the moving frame, then,

$$P' = \frac{\mathrm{d}W'}{\mathrm{d}t'} = \frac{2q^2}{3c^3}\left|\vec{a}'\right|^2 = \frac{2q^2}{3c^3}\left({a'_\parallel}^2 + {a'_\perp}^2\right) \tag{4}$$

and, transforming to the lab frame,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}W'}{\mathrm{d}t'} \tag{5}$$

if there is no net momentum in the primed frame.

$$a_\parallel = 0 \tag{6}$$

$$a_\perp = \omega_B v_\perp = \frac{qBv\sin\alpha}{\gamma m c} \tag{7}$$

where $a_\parallel$ and $a_\perp$ are parallel and perpendicular, respectively, to $\vec{\beta}$, $v_\perp$ is perpendicular to $\vec{B}$, and α is the pitch angle. After the transformation of accelerations,

$$a_\parallel = \frac{1}{\gamma^3} a'_\parallel \tag{8}$$

$$a_\perp = \frac{1}{\gamma^2} a'_\perp, \tag{9}$$

we find

$$P = \frac{2}{3} \frac{q^4}{m^2 c^3} \gamma^2 \beta^2 B^2 \sin^2 \alpha. \tag{10}$$

However, $\langle \sin^2 \alpha \rangle = \frac{2}{3}$, and we can substitute $U_B = \frac{1}{8\pi} B^2$ as the energy of the magnetic field, and we arrive at the common formula for electrons:

$$P = \frac{4}{3} n \sigma_T c \gamma^2 \beta^2 U_B \tag{11}$$

This assumes an electron density n and an isotropic distribution.

2 Synchrotron spectrum of a single electron

We consider a simple magnetic field $\vec{B} = B\hat{z}$. In this field, an electron moves relativistically with frequency

$$\omega_B = \frac{qB}{\gamma m c} = \frac{eB}{\gamma m_e c} \tag{12}$$

as shown in Figure 1. Because it moves relativistically, its radiation is beamed into a small cone in the forward direction. The particle radiates towards the observer only over an angle range $\sim 2/\gamma$ during a full orbit. The electric field at the observer looks like that shown in Figure 2.

The spectral power from a single electron is given by

$$P(\omega)\, \mathrm{d}\omega = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin \alpha}{m c^2} F(x)\, \mathrm{d}\omega \tag{13}$$

where $x \equiv \omega/\omega_c$,

$$\omega_c \equiv \frac{3}{2} \gamma^3 \omega_B \sin \alpha, \tag{14}$$

and

$$F(x) = x \int_x^\infty K_{5/3}(\xi)\, \mathrm{d}\xi. \tag{15}$$

Note that the power is just a complicated coefficient multiplied by a dimensionless function, so all the information about the spectral shape is in $F(x)$. In the extreme limits,

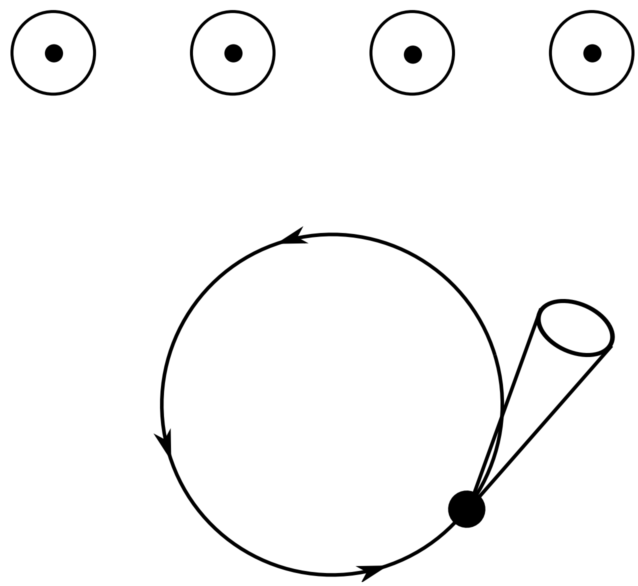

Figure 1: Single electron following a helical path in a magnetic field coming out of the page. Because the particle move relativistically, its radiation is beamed forward as shown.

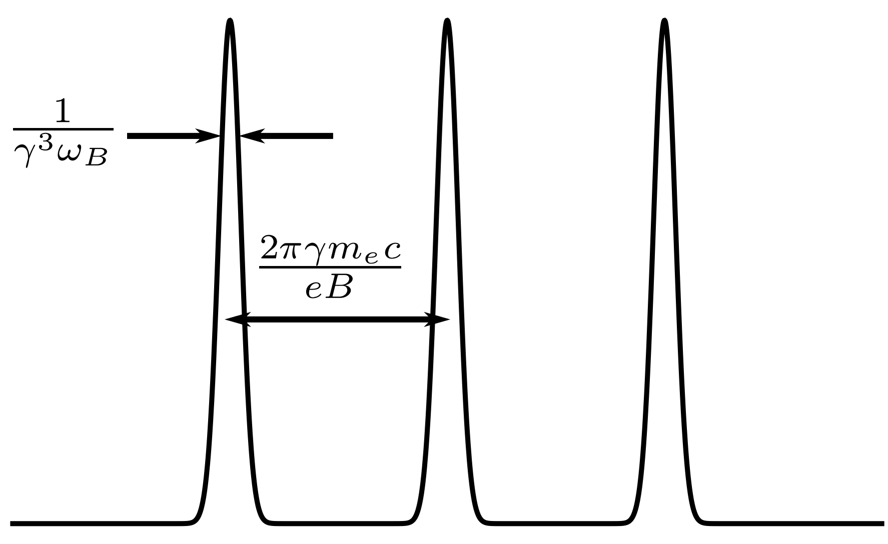

Figure 2: Electric field at an observer due to the synchrotron emission of a single electron.

$$F(x) \approx \begin{cases} \left[\frac{4\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}\right]\left(\frac{x}{2}\right)^{1/3}, & x \ll 1 \\ \left(\frac{\pi}{2}\right)^{1/2} x^{1/2} e^{-x}, & x \gg 1 \end{cases}. \tag{16}$$

So the power spectrum of a single electron is $\sim \omega^{1/3}$ when $\omega \ll \omega_c$ and $\sim e^{-x}$ when $\omega \gg \omega_c$, with a turnover occurring when $\omega = \omega_c$.

3 Synchrotron cooling

The particle energy is always given by $E = \gamma mc^2$. Additionally,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -P_{\text{synch}}; \tag{17}$$

that is, the change in energy is proportional to the power. So we can write

$$mc^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{4}{9}\frac{q^2}{m^2c^3}B^2\gamma^2. \tag{18}$$

If, at time $t = 0$, $\gamma = \gamma_i$ is the initial Lorentz factor,

$$\frac{1}{\gamma(t)} = \frac{1}{\gamma_i} + \frac{t}{t_s}, \tag{19}$$

where t_s is some characteristic timescale,

$$t_s = \frac{\gamma mc^2}{P} = \frac{9}{4}\frac{\left(mc^2\right)^4}{Eq^4B^2c}. \tag{20}$$

Assume $\gamma_i \sim \infty$, so $1/\gamma_i \approx 0$. Then, for electrons,

$$\gamma(t) \approx \left(2.5 \times 10^{13}\right)\left(\frac{B}{1\ \mu\text{G}}\right)^{-2}\left(\frac{t}{1\ \text{yr}}\right)^{-1} \tag{21}$$

for electrons. The system basically does not care about initial conditions, just on the amount of time elapsed! Using the formulæfrom the previous sections, we can sketch the evolution of a synchrotron spectrum over time, as in Figure 3.

4 Radiometer equation

We return to the diagram of an antenna with receiver, etc., shown in Figure 4. Based on the derivation in the last set of notes,

$$\sigma_3^2 = \frac{\left[Gk\left(T_A + T_{\text{rx}}\right)\right]^2 \Delta\nu}{\Delta\nu\tau} \tag{22}$$

This leads to the “ideal radiometer equation,”

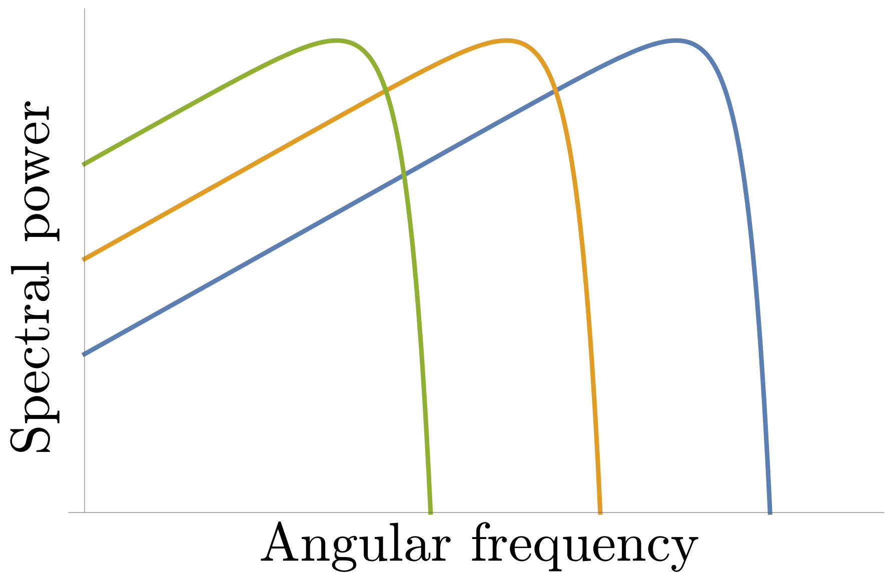

Figure 3: Cooling of a synchrotron spectrum.

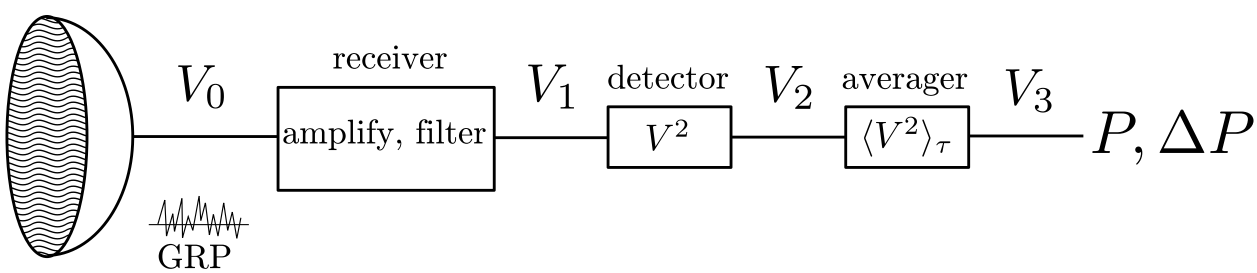

Figure 4: Diagram of antenna, receiver, and detector.

$$\Delta T = \frac{T_{\rm rx} + T_A}{\sqrt{\tau \Delta\nu}}, \tag{23}$$

where ΔT is the variation in measured temperature. It is the lower limit for integrated noise, but note that there are many noise sources that could add to this!

5 Friis equation

Consider a signal path that includes multiple components, each with its own intrinsic noise temperature and gain. The Friis equation (*not* the Friis transmission equation) tells us that

$$T_{\rm noise} = T_{\rm rx1} + T_{\rm rx2}/G_1 + T_{\rm rx3}/G_1 G_2 + \cdots \tag{24}$$

where G_n is the gain of the nth component and $T_{{\rm rx}n}$ is its noise temperature.

6 Y factor

The Y factor is quite simple: it is just a power ratio:

$$Y = \frac{T_{\rm rx} + T_{\rm hot}}{T_{\rm rx} + T_{\rm cold}} \tag{25}$$

It gives some measure of "quality" of your receiver, since $Y \to \infty$ is a very good receiver. You should watch out for systematic effects that tend to reduce Y.